Explaining Global Constraints from their Decompositions

Arthur Gontier, Charlotte Truchet, Charles Prud'homme
LS2N, IMT Atlantique, Université de Nantes

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CP model

Model

Variables: x, y, z

Constraints: $c1, c2, c3$

Domains: D_x, D_y, D_z

Solving

Filtering algorithms

Search Strategies

Conflict Driven Clause Learning

CP CDCL (HaifaCSP)

Lazy Clause Generation (Chuffed)

Motivations for CDCL in CP

Pros

- Reduce search space (rcpsp)
- Help strategies
- Proof of failure

From SAT to CP

- Boolean variable

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Boolean variable + Integer variable

Boolean clauses + a lot of different global constraints

Problem: Each constraint needs an explanation

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Boolean variable + Integer variable

Boolean clauses + a lot of different global constraints

Problem: Each constraint needs an explanation

Generate explanation?

From decomposition to explanation

Table of content

Section summary

CP solving: Implication graph

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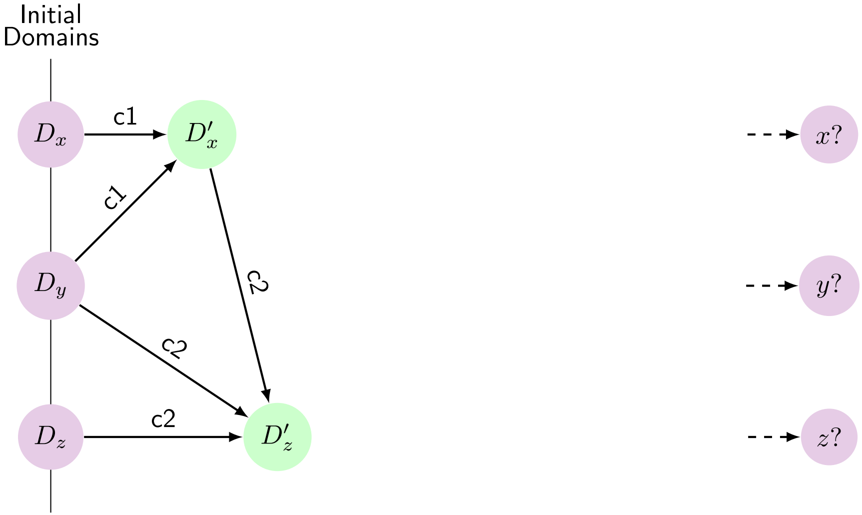
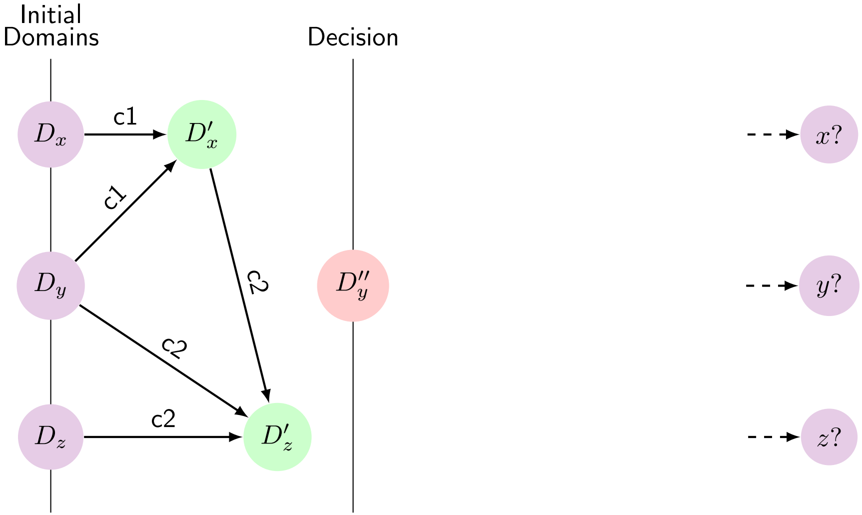
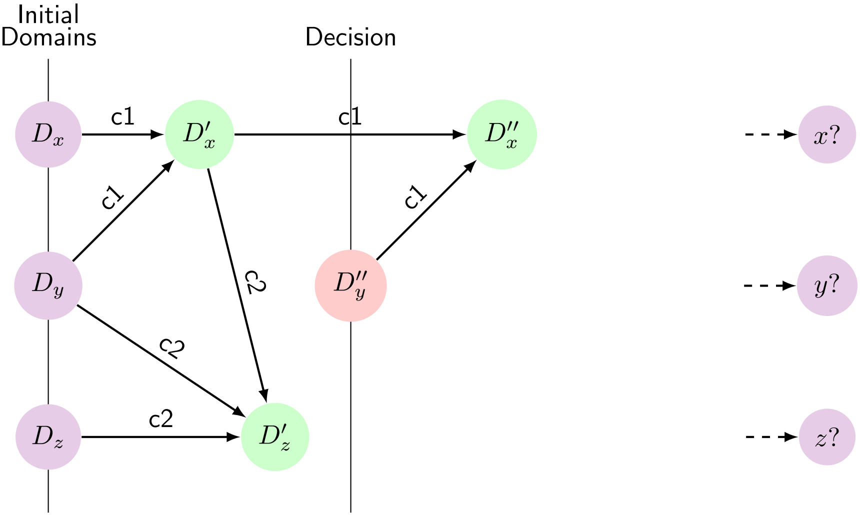


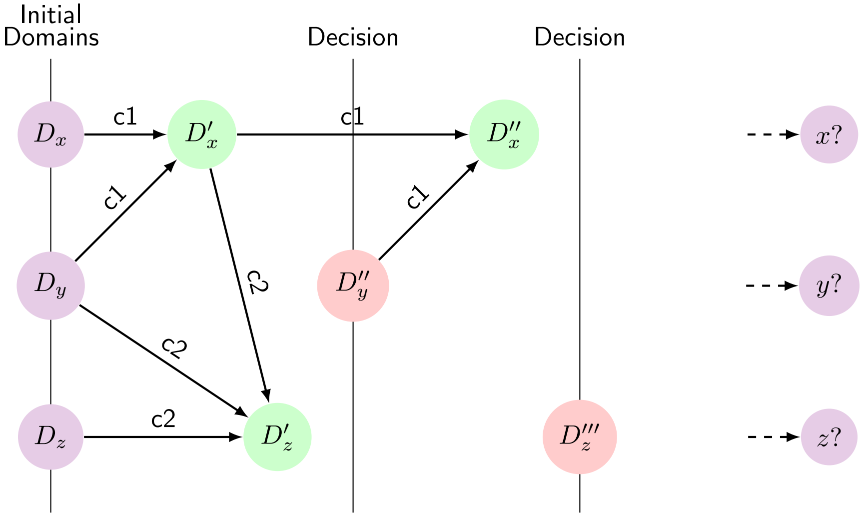
CP solving: Implication graph



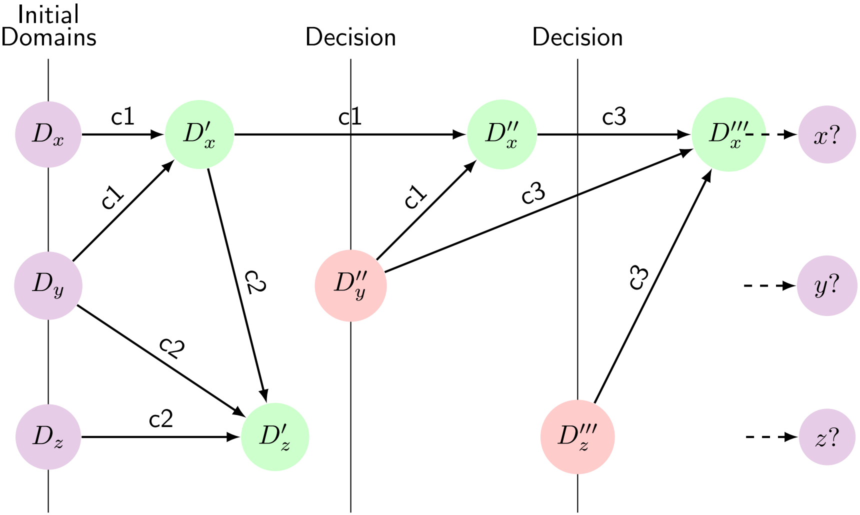
CP solving: Implication graph



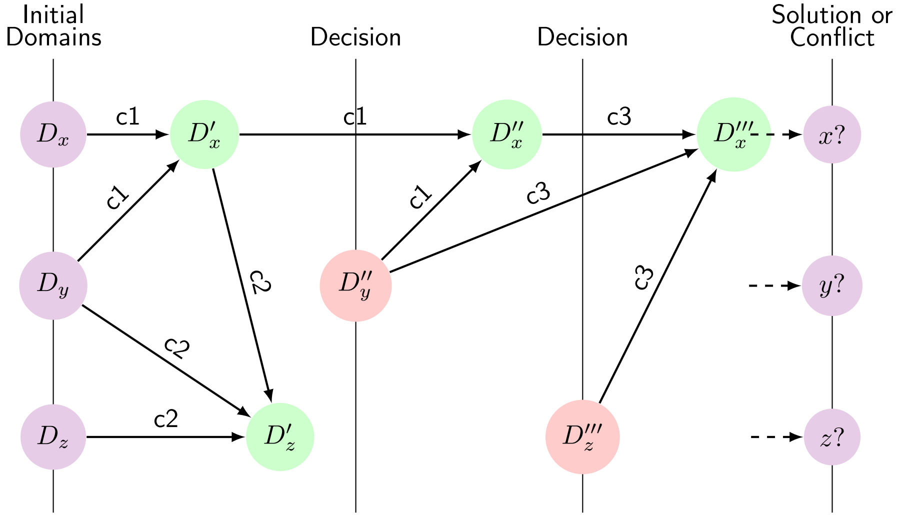
CP solving: Implication graph



CP solving: Implication graph



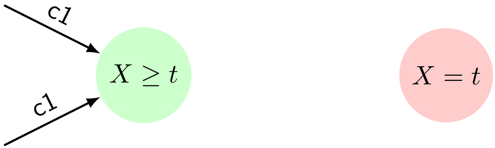
CP solving: Implication graph



Events

Definition (Event and explanation)

An event is a domain reduction written ($X \leq t$, $X \geq t$, $X \neq t$, $X = t$) in the following. It is caused by either propagation or decision.



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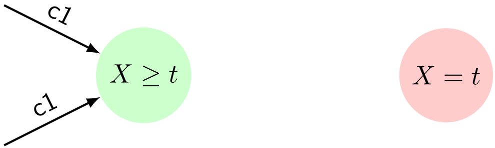


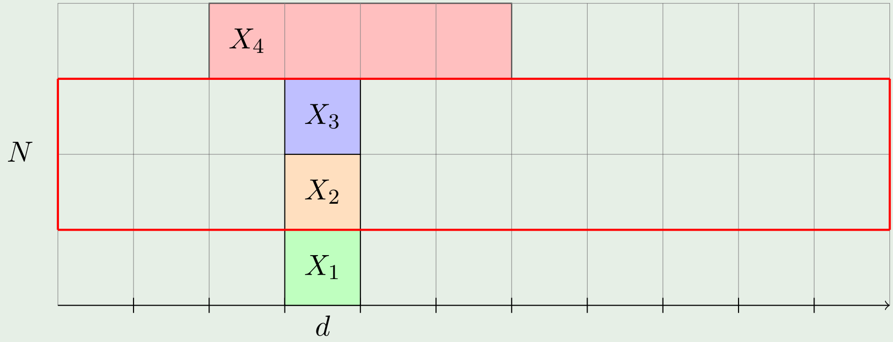
Definition (Explanation)

An explanation is a conjunction of events implying an event generated by a constraint propagation. In the following, it is noted:

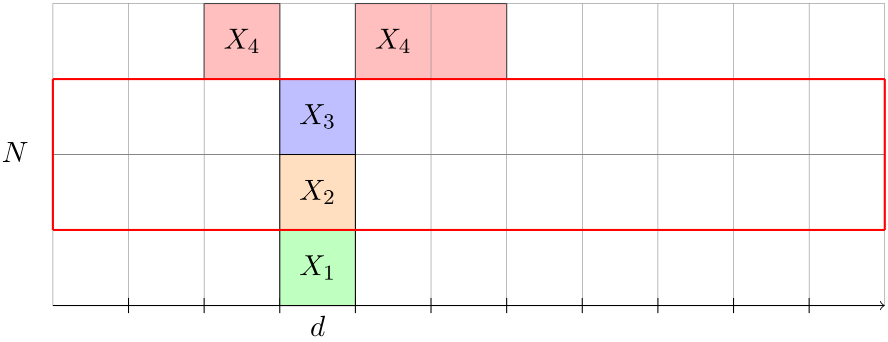
$$\langle explaining\ events \rangle \rightarrow \langle event \rangle$$

Propagation and decision

AtMost$(N, \{X_1, \ldots, X_4\}, d)$



Explanation



$$\langle N < 4\rangle\langle X_1 = d\rangle\langle X_2 = d\rangle\langle X_3 = d\rangle \rightarrow \langle X_4 \neq d\rangle$$

GOAL

Find explanation rules for each constraint

Section summary

Decomposition language

Definition (Reification)

A reified constraint c is associated to a Boolean variable b such that the truth state of b matches the satisfaction state of c.

$$Constraint \iff b$$

Decomposition language

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$$\begin{cases} \text{Equality: } X_i = t \iff b_{it} & \forall i \in [\![1, n]\!]\ \forall t \in [\![1, m]\!] \\ \text{Inequality: } X_i \geq t \iff b_{it} & \forall i \in [\![1, n]\!]\ \forall t \in [\![1, m]\!] \end{cases}$$

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Constraint decomposition

Atmost(t,X,v)

$$\begin{cases} x_i = v \iff b_i, & \forall i \in [1, |X|] \\ \sum_{i \in [1,|X|]} b_i \leq t \end{cases}$$

To generate an explanation rule we need:

- A rule for each event of the decomposition language
- A rewriting algorithm

Rewriting rules

$$Equality:$$

$$\langle x_i = t\rangle \xrightarrow{R_=} \langle b_{it}\rangle$$

$$\langle b_{it}\rangle \xrightarrow{R_=} \langle x_i = t\rangle$$

$$\langle x_i \neq t\rangle \xrightarrow{R_{\neq}} \langle \neg b_{it}\rangle$$

$$\langle \neg b_{it}\rangle \xrightarrow{R_{\neq}} \langle x_i \neq t\rangle$$

$$Inequality:$$

$$\langle x_i \leq t\rangle \xrightarrow{R_{\leq}} \langle b_{it}\rangle$$

$$\langle b_{it}\rangle \xrightarrow{R_{\leq}} \langle x_i \leq t\rangle$$

$$\langle x_i > t\rangle \xrightarrow{R_{>}} \langle \neg b_{it}\rangle$$

$$\langle \neg b_{it}\rangle \xrightarrow{R_{>}} \langle x_i > t\rangle$$

Rewriting rules

Sum

$$\langle b_i \rangle \xrightarrow{R^1_\Sigma} \langle \neg b \rangle \langle \neg b_j \rangle_{\forall j \neq i}$$

$$\langle \neg b_i \rangle \xrightarrow{R^2_\Sigma} \langle b \rangle \langle b_j \rangle_{\forall j \neq i}$$

$$\langle b \rangle \xrightarrow{R^3_\Sigma} \langle \neg b_i \rangle_{\forall i}$$

$$\langle \neg b \rangle \xrightarrow{R^4_\Sigma} \langle b_i \rangle_{\forall i}$$

And

$$\langle b_i \rangle \xrightarrow{R^1_\wedge} \langle b_i \rangle$$

$$\langle \neg b_i \rangle \xrightarrow{R^2_\wedge} \langle \neg b \rangle \langle b_j \rangle_{\forall j \neq i}$$

$$\langle b \rangle \xrightarrow{R^3_\wedge} \langle b_i \rangle_{\forall i}$$

$$\langle \neg b \rangle \xrightarrow{R^4_\wedge} \langle \neg b_i \rangle_{\exists i}$$

Or

$$\langle b_i \rangle \xrightarrow{R^1_\vee} \langle b \rangle \langle \neg b_j \rangle_{\forall j \neq i}$$

$$\langle \neg b_i \rangle \xrightarrow{R^2_\vee} \langle \neg b \rangle$$

$$\langle b \rangle \xrightarrow{R^3_\vee} \langle b_i \rangle_{\exists i}$$

$$\langle \neg b \rangle \xrightarrow{R^4_\vee} \langle \neg b_i \rangle_{\forall i}$$

Example: AtMost= $\begin{cases} x_i = v \iff b_i \\ \sum_i b_i \leq t \end{cases}$

Constraints with term: $X_i = t$

$$x_i = v \iff b_i$$

Example: AtMost= $\begin{cases} x_i = v \iff b_i \\ \sum_i b_i \leq t \end{cases}$

Constraints with term: $X_i = t$

$$x_i = v \iff b_i$$

Possible rules

$$\langle x_i = t \rangle \xrightarrow{R_=} \langle b_i \rangle$$

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Constraints with term: $X_i = t$

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Possible rules

$$\langle x_i = t \rangle \xrightarrow{R_=} \langle b_i \rangle$$

Formula

$$b_i$$

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Constraints with term: b_i

$$x_i = v \iff b_i$$

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Constraints with term: b_i

$$x_i = v \iff b_i$$

$$\sum_i b_i \leq t$$

Possible rules

$$\langle b_{it} \rangle \xrightarrow{R_=} \langle x_i = t \rangle$$

$$\langle b_i \rangle \xrightarrow{R^1_\Sigma} \langle \neg b \rangle \langle \neg b_j \rangle_{\forall j \neq i}$$

Example: AtMost= $\begin{cases} x_i = v \iff b_i \\ \sum_i b_i \leq t \end{cases}$

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Formula

$$\bot \wedge \neg b_j \quad \forall j \neq i$$

Explanation

$$\langle \bot \rangle \rightarrow \langle X_i = v \rangle$$

Example: AtMost= $\begin{cases} x_i = v \iff b_i \\ \sum_i b_i \leq t \end{cases}$

Constraints with term: $X_i \neq v$

$$x_i = v \iff b_i$$

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Constraints with term: $X_i \neq v$

$$x_i = v \iff b_i$$

Possible rules

$$\langle x_i \neq t \rangle \xrightarrow{R_{\neq}} \langle \neg b_i \rangle$$

Example: AtMost= $\begin{cases} x_i = v \iff b_i \\ \sum_i b_i \leq t \end{cases}$

Constraints with term: $X_i \neq v$

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Possible rules

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Formula

$$\neg b_i$$

Example: AtMost= $\begin{cases} x_i = v \iff b_i \\ \sum_i b_i \le t \end{cases}$

Constraints with term: $\neg b_i$

$$x_i = v \iff b_i$$

$$\sum_i b_i \le t$$

Example: AtMost$=\begin{cases} x_i = v \iff b_i \\ \sum_i b_i \leq t \end{cases}$

Constraints with term: $\neg b_i$

$$x_i = v \iff b_i$$

$$\sum_i b_i \leq t$$

Possible rules

$$\langle \neg b_{it} \rangle \xrightarrow{R_{\neq}} \langle x_i \neq t \rangle$$

$$\langle \neg b_i \rangle \xrightarrow{R^2_{\Sigma}} \langle b \rangle \langle b_j \rangle_{\forall j \neq i}$$

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Formula

$$\top \wedge b_j \quad \forall j \neq i$$

Example: AtMost= $\begin{cases} x_i = v \iff b_i \\ \sum_i b_i \leq t \end{cases}$

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$$\langle x_i = t \rangle \xrightarrow{R_=} \langle b_i \rangle$$

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$$\langle b_i \rangle \xrightarrow{R^1_\Sigma} \langle \neg b \rangle \langle \neg b_j \rangle_{\forall j \neq i}$$

Formula

$$\top \wedge X_j = v \quad \forall j \neq i$$

Explanation

$$\langle X_j = v \rangle_{\forall j \neq i} \rightarrow \langle X_i \neq v \rangle$$

Results

Two new explained constraints in Chuffed

- Count
- Increasing

instance	explanation	nodes	fails	backjumps	time (ms)
league (model15-4-3)	chuffed	10487	9717	639	393
	default	51383	51067	187	2641
	generated	4501	4038	330	189
oocsp (racks_030_mii8)	chuffed	3724887	3616895	107972	179025
	default	3951358	3875074	76263	219893
	generated	3807940	3713037	94882	186959
oc-rooster (4s-23d)	chuffed	71044	54342	16367	4432
	default	792690	774296	18055	24617
	generated	82421	62145	19939	5530

Table: Three example instances comparing `chuffed`, the default explanation and the generated explanations.

Section summary

Conclusion

In this talk:

- A simple decomposition language
- A rewriting rule algorithm to generate an explanation
- An implementation in Chuffed

Conclusion

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- A simple decomposition language
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Future work:

- Expand the decomposition language (rule learning?)
- aglorithmically global constraints decompositions ?
- What to do when there is more than one decompositions?

Thanks !

Algorithms: Index modification and propagation

Definition (Index modification)

$i \diamond i', \diamond \in \{=, \neq, \geq, \leq\}$: a relation between two indexes
$i \diamond n, \diamond \in \{+, -\}, n \in \mathbb{N}$: a modification by an integer
$i \in D$: a belonging to a set D
$\exists i$ and $\forall i$: a way to introduce indexes

Index propagation in the graph

index_propagate: propagate the index as described in the equation
index_update: synchronise the graph index with the equation one

Example $b_{i(t-d_i)}$

index_propagate: $(i, t) \rightarrow (i, t - d_i)$
index_update: $(i, t) \rightarrow (i, t + d_i)$

CDCL applyed to CP (Veksler & Strichman)

CSP-Analyze-Conflict

```
cl←Explain(conflict-node)
pred←Predecessors(conflict-node)
front←Relvant(pred,cl)
while ¬ Stop-criterion-met(front) do
    curr-node←Last-node(front)
    front←front⊂ curr-node
    expl ← Explain(curr-node)
    cl←Resolve(cl,expl,var(lit(curr-node)))
    pred←Predecessors(curr-node)
    front←Distinct(Relvant(front∪pred,cl))
add-clause-to-database(cl)
```

Generated rule: Choco

Instance	explainless	generated explain	Chuffed
rcpsp-02	0.076s 86 Nodes 83 Fails	0.071s 124 Nodes 0 Fails	0.041s 125 Nodes 1 Fails
rcpsp-03	0.095s 62 Nodes 61 Fails	0.072s 62 Nodes 0 Fails	0.040s 62 Nodes 0 Fails
rcpsp-04	>1h	>1h	885.184s 4049049 Nodes 3153463 Fails
rcpsp-05	>3m 6870746 Nodes 6870673 Fails	0.409s 1224 Nodes 78 Fails	0.159s 2271 Nodes 102 Fails

Generated rule: Choco

Instance	explainless	generated explain	Chuffed
rcpsp-06	0.151s 186 Nodes 177 Fails	0.329s 445 Nodes 0 Fails	0.140s 446 Nodes 1 Fails
rcpsp-07	0.708s 305 Nodes 296 Fails	0.680s 422 Nodes 0 Fails	0.259s 422 Nodes 0 Fails
rcpsp-08	>1h	>1h	>1h
rcpsp-09	0.722s 577 Nodes 546 Fails	0.535s 1832 Nodes 10 Fails	0.268s 2275 Nodes 15 Fails

AllDifferent($\{X_1, \ldots, X_n\}$)

$$X_i = t \iff b_{it} \qquad \forall i \in [\![1, n]\!] \forall t \in [\![1, m]\!]$$

$$\sum_{i \in [\![1,n]\!]} b_{it} \leq 1 \qquad \forall t \in [\![1, m]\!]$$

$$\frac{X_{i'} = t,\ \forall i',\ i' \neq i,\ i' \in [\![1, n]\!],\ i \in [\![1, n]\!]}{X_i \neq t}$$

AllEqual($\{X_1, \ldots, X_n\}$)

$$X_i \geq t \iff b_{it} \qquad \forall i \in [\![1, n]\!] \forall t \in [\![1, m]\!]$$

$$\bigwedge_{i \in [\![1,n]\!]} b_{it} \iff b1_t \qquad \forall t \in [\![1, m]\!]$$

$$\bigwedge_{i \in [\![1,n]\!]} \neg b_{it} \iff b2_t \qquad \forall t \in [\![1, m]\!]$$

$$(b1_t \vee b2_t) \qquad \forall t \in [\![1, m]\!]$$

$$\frac{X_i \geq t,\ \exists i,\ i \in [\![1, n]\!]}{X_i \geq t}$$

$$\frac{X_{i'} < t,\ \forall i',\ i' \neq i,\ i' \in [\![1, n]\!],\ i \in [\![1, n]\!]}{X_i \geq t}$$

$$\frac{X_{i'} \geq t,\ \forall i',\ i' \neq i,\ i' \in [\![1, n]\!],\ i \in [\![1, n]\!]}{X_i < t}$$

$$\frac{X_i < t,\ \exists i,\ i \in [\![1, n]\!]}{X_i < t}$$

NValue$(N, \{X_1, \ldots, X_n\})$

$$
\begin{aligned}
X_i = t &\iff b_{it} && \forall i \in [\![1,n]\!] \forall t \in [\![1,m]\!] \\
\bigvee_{i \in [\![1,n]\!]} b_{it} &\iff b2_t && \forall t \in [\![1,m]\!] \\
\sum_{t \in [\![1,m]\!]} b2_t = p &\iff b3_p && \forall t \in [\![1,m]\!] \forall p \in [\![1,n]\!] \\
N = p &\iff b3_p && \forall p \in [\![1,n]\!]
\end{aligned}
$$

$$\frac{X_i \neq t,\ \forall i,\ i \in [\![1,n]\!],\ \forall t,\ t \in [\![1,m]\!] \quad N = p,\ \forall p,\ p \in [\![1,n]\!]}{X_i = t}$$

$$\frac{X_{i'} \neq t,\ \forall i',\ i' \neq i,\ i' \in [\![1,n]\!],\ i \in [\![1,n]\!]}{X_i = t}$$

$$\frac{X_i = t,\ \exists i,\ i \in [\![1,n]\!],\ \forall t,\ t \in [\![1,m]\!] \quad N = p,\ \forall p,\ p \in [\![1,n]\!]}{X_i \neq t}$$

$$\frac{X_i \neq t,\ \forall i,\ i \in [\![1,n]\!],\ \forall t,\ t \in [\![1,m]\!],\ \forall i,\ i \in [\![1,n]\!] \quad X_i = t,\ \exists i,\ i \in [\![1,n]\!],\ \forall t,\ t \in [\![1,m]\!],\ \forall i,\ i \in [\![1,n]\!]}{N = p}$$

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AtLeastNValue($N, \{X_1, \dots, X_n\}$)

$$X_i = t \iff b_{it} \qquad \forall i \in [\![1, n]\!] \forall t \in [\![1, m]\!]$$

$$\bigvee_{i \in [\![1,n]\!]} b_{it} \iff b2_t \qquad \forall t \in [\![1, m]\!]$$

$$\sum_{t \in [\![1,m]\!]} b2_t \geq p \iff b3_p \qquad \forall t \in [\![1, m]\!]$$

$$N \geq p \iff b3_p \qquad \forall t \in [\![1, m]\!]$$

$$\frac{X_i \neq t',\ \forall i,\ i \in [\![1, n]\!],\ \forall i,\ i \in [\![1, n]\!],\ \forall t',\ t' \neq t,\ t' \in [\![1, m]\!],\ t \in [\![1, m]\!] \quad N \geq p,\ \forall p,\ p \in [\![1, n]\!]}{X_i = t}$$

$$\frac{X_{i'} \neq t,\ \forall i',\ i' \neq i,\ i' \in [\![1, n]\!],\ i \in [\![1, n]\!]}{X_i = t}$$

$$\frac{X_i = t',\ \exists i,\ i \in [\![1, n]\!],\ \forall i,\ i \in [\![1, n]\!],\ \forall t',\ t' \neq t,\ t' \in [\![1, m]\!],\ t \in [\![1, m]\!] \quad N < p,\ \forall p,\ p \in [\![1, n]\!]}{X_i \neq t}$$

$$\frac{X_i = t,\ \exists i,\ i \in [\![1, n]\!],\ \forall i,\ i \in [\![1, n]\!],\ \forall t,\ t \in [\![1, m]\!],\ \forall i,\ i \in [\![1, n]\!]}{N \geq p}$$

$$\frac{X_i \neq t,\ \forall i,\ i \in [\![1, n]\!],\ \forall i,\ i \in [\![1, n]\!],\ \forall t,\ t \in [\![1, m]\!],\ \forall i,\ i \in [\![1, n]\!]}{N < p}$$

AtMostNValue($N, \{X_1, \dots, X_n\}$)

$$X_i = t \iff b_{it} \qquad \forall i \in [\![1, n]\!] \forall t \in [\![1, m]\!]$$

$$\bigvee_{i \in [\![1,n]\!]} b_{it} \iff b2_t \qquad \forall t \in [\![1, m]\!]$$

$$\sum_{t \in [\![1,m]\!]} b2_t < p \iff \neg b3_p \qquad \forall t \in [\![1, m]\!]$$

$$N \geq p \iff b3_p \qquad \forall t \in [\![1, m]\!]$$

$$\frac{X_i \neq t',\ \forall i,\ i \in [\![1, n]\!],\ \forall i,\ i \in [\![1, n]\!],\ \forall t',\ t' \neq t,\ t' \in [\![1, m]\!],\ t \in [\![1, m]\!] \quad N \geq p,\ \forall p,\ p \in [\![1, n]\!]}{X_i = t}$$

$$\frac{X_{i'} \neq t,\ \forall i',\ i' \neq i,\ i' \in [\![1, n]\!],\ i \in [\![1, n]\!]}{X_i = t}$$

$$\frac{X_i = t',\ \exists i,\ i \in [\![1, n]\!],\ \forall i,\ i \in [\![1, n]\!],\ \forall t',\ t' \neq t,\ t' \in [\![1, m]\!],\ t \in [\![1, m]\!] \quad N < p,\ \forall p,\ p \in [\![1, n]\!]}{X_i \neq t}$$

$$\frac{X_i = t,\ \exists i,\ i \in [\![1, n]\!],\ \forall i,\ i \in [\![1, n]\!],\ \forall t,\ t \in [\![1, m]\!],\ \forall i,\ i \in [\![1, n]\!]}{N \geq p}$$

$$\frac{X_i \neq t,\ \forall i,\ i \in [\![1, n]\!],\ \forall i,\ i \in [\![1, n]\!],\ \forall t,\ t \in [\![1, m]\!],\ \forall i,\ i \in [\![1, n]\!]}{N < p}$$

Cumulative$(\{X_1, \ldots, X_n\}, \{d_1, \ldots, d_n\}, c)$

$$X_i \geq t \iff b_{it} \qquad \forall i \in [\![1,n]\!] \forall t \in [\![1,m]\!]$$

$$(b_{i(t-d_i)} \wedge \neg b_{it}) \iff b2_{it} \qquad \forall i \in [\![1,n]\!] \forall t \in [\![1,m]\!]$$

$$\sum_{i \in [\![1,n]\!]} b2_{it} \leq c \qquad \forall t \in [\![1,m]\!]$$

$$\frac{X_i \geq t',\ t' = t - d_i \qquad X_{i'} < t,\ \forall i' \neq i,\ i' \in [\![1,n]\!] \qquad X_{i'} \geq t',\ t' = t - d_{i'},\ \forall i' \neq i,\ i' \in [\![1,n]\!]}{X_i \geq t}$$

$$\frac{X_i < t',\ t' = t + d_i \qquad X_{i'} < t',\ t' = t + d_i,\ \forall i' \neq i,\ i' \in [\![1,n]\!] \qquad X_{i'} \geq t'',\ t'' = t' - d_{i'},\ t' = t + d_i,\ \forall i' \neq i,\ i' \in [\![1,n]\!]}{X_i < t}$$

Element$(I, \{X_1, \ldots, X_n\}, V)$

$$X_i = t \iff b_{it}^X \qquad \forall i \in [\![1, n]\!] \forall t \in [\![1, m]\!]$$

$$I = i \iff b_i^I \qquad \forall i \in [\![1, n]\!]$$

$$V = t \iff b_t^V \qquad \forall t \in [\![1, m]\!]$$

$$\neg b_t^V \wedge \neg b_i^I \wedge b_{it}^X \qquad \forall i \in [\![1, n]\!] \forall t \in [\![1, m]\!]$$

$$b_t^V \wedge \neg b_i^I \wedge \neg b_{it}^X \qquad \forall i \in [\![1, n]\!] \forall t \in [\![1, m]\!]$$

$$\frac{I = i \quad V = t}{X_i = t}$$

$$\frac{I = i \quad V \neq t}{X_i \neq t}$$

$$\frac{X_i \neq t,\ \forall t,\ t \in [\![1, m]\!] \quad V = t,\ \forall t,\ t \in [\![1, m]\!]}{I \neq i}$$

$$\frac{X_i = t,\ \forall t,\ t \in [\![1, m]\!] \quad V \neq t,\ \forall t,\ t \in [\![1, m]\!]}{I \neq i}$$

$$\frac{X_i = t,\ \forall i,\ i \in [\![1, n]\!] \quad I = i,\ \forall i,\ i \in [\![1, n]\!]}{V = t}$$

Gcc($\{X_1, \ldots, X_n\}, V, O$)

$$X_i = t \iff b_{it} \qquad \forall i \in [\![1, n]\!] \forall t \in [\![1, m]\!]$$

$$O_t = p \iff b2_{tp} \qquad \forall t \in [\![1, m]\!] \forall p \in [\![1, n]\!]$$

$$\sum_{i \in [\![1,n]\!]} b_{it} \geq p \iff b2_{tp} \qquad \forall t \in [\![1, m]\!] \forall p \in [\![1, n]\!]$$

$$\frac{X_{i'} \neq t,\ \forall i',\ i' \neq i,\ i' \in [\![1, n]\!],\ i \in [\![1, n]\!] \quad O_t \geq p,\ \forall p,\ p \in [\![1, n]\!]}{X_i = t}$$

$$\frac{X_{i'} = t,\ \forall i',\ i' \neq i,\ i' \in [\![1, n]\!],\ i \in [\![1, n]\!] \quad O_t < p,\ \forall p,\ p \in [\![1, n]\!]}{X_i \neq t}$$

$$\frac{X_i = t,\ \forall i,\ i \in [\![1, n]\!],\ \forall i,\ i \in [\![1, n]\!]}{O_t \geq p}$$

$$\frac{X_i \neq t,\ \forall i,\ i \in [\![1, n]\!],\ \forall i,\ i \in [\![1, n]\!]}{O_t < p}$$

Increasing($\{X_1, \ldots, X_n\}$)

$$X_i \geq t \iff b_{it} \qquad \forall i \in [\![1, n]\!] \forall t \in [\![1, m]\!]$$

$$(\neg b_{(i-1)t} \vee b_{it}) \qquad \forall i \in [\![2, n]\!] \forall t \in [\![1, m]\!]$$

$$\frac{X_{i'} \geq t,\ i' = i - 1}{X_i \geq t}$$

$$\frac{X_{i'} < t,\ i' = i + 1}{X_i < t}$$

Decreasing($\{X_1, \ldots, X_n\}$)

$$X_i \geq t \iff b_{it} \qquad \forall i \in [\![1, n]\!] \forall t \in [\![1, m]\!]$$

$$(b_{(i-1)t} \vee \neg b_{it}) \qquad \forall i \in [\![2, n]\!] \forall t \in [\![1, m]\!]$$

$$\frac{X_{i'} \geq t, \;\; i' = i + 1}{X_i \geq t}$$

$$\frac{X_{i'} < t, \;\; i' = i - 1}{X_i < t}$$

Among$(c, \{X_1, \ldots, X_n\}, D_4)$

$$X_i = t \iff b_{it} \qquad \forall i \in [\![1, n]\!] \forall t \in [\![1, m]\!]$$

$$\bigvee_{t \in D_4} b_{it} \iff b2_i \qquad \forall i \in [\![1, n]\!]$$

$$\sum_{i \in [\![1,n]\!]} b2_i = c$$

$$\frac{X_i \neq t,\ \forall t,\ t \in D_4,\ \forall i,\ i \in [\![1, n]\!]}{X_i = t}$$

$$\frac{X_i \neq t',\ \forall t',\ t' \neq t,\ t' \in D_4,\ t \in D_4}{X_i = t}$$

$$\frac{X_i = t,\ \exists t,\ t \in D_4,\ \forall i,\ i \in [\![1, n]\!]}{X_i \neq t}$$

Roots($\{X_1, \ldots, X_n\}, I, V$)

$$X_i = t \iff b_{it} \qquad \forall it$$

$$\sum_{t \in V} b_{it} = 1 \qquad \forall i \in I$$

$$\sum_{t \in D \setminus V} b_{it} = 1 \qquad \forall i \in [\![1, n]\!] \setminus I$$

$$\frac{X_i \neq t,\ \forall t,\ t \in D_5}{X_i = t}$$

$$\frac{X_i \neq t,\ \forall t,\ t \in D_6}{X_i = t}$$

$$\frac{X_i = t,\ \forall t,\ t \in D_5}{X_i \neq t}$$

$$\frac{X_i = t,\ \forall t,\ t \in D_6}{X_i \neq t}$$

$V = D_5$ and $D \setminus V = D_6$

Range$(\{X_1, \ldots, X_n\}, I, V)$

$$X_i = t \iff b_{it} \qquad \forall it$$

$$\sum_{i \in I} b_{it} \geq 1 \qquad \forall t \in V$$

$$\sum_{t \in V} b_{it} = 1 \qquad \forall i \in I$$

$$\frac{X_i \neq t',\ \forall t',\ t' \neq t,\ t' \in D_5,\ t \in D_5}{X_i = t}$$

$$\frac{X_i \neq t,\ \forall t,\ t \in D_6}{X_i = t}$$

$$\frac{X_i = t,\ \forall t,\ t \in D_6}{X_i \neq t}$$

$I = D_5$ and $V = D_6$

Example: Cumulative

- X_i: Start time of the ith task
- d_i: Duration of the ith task
- c: Resource capacity

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- d_i: Duration of the ith task
- c: Resource capacity

Cumulative$(\{X_1, \ldots, X_n\}, \{d_1, \ldots, d_n\}, c) \iff (1) \wedge (2) \wedge (3)$

$$X_i \geq t \iff b_{it} \qquad \forall i \in [\![1, n]\!] \forall t \in [\![1, m]\!] \qquad (1)$$

$$(b_{i(t-d_i)} \wedge \neg b_{it}) \iff b2_{it} \qquad \forall i \in [\![1, n]\!] \forall t \in [\![1, m]\!] \qquad (2)$$

$$\sum_{i \in [\![1,n]\!]} b2_{it} \leq c \qquad \forall t \in [\![1, m]\!] \qquad (3)$$

Example: Cumulative

Generated event: lower bound update

$$X_i \geq t$$

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Constraints with this event

$$X_i \geq t \iff b_{it}$$

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Constraints with this event

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Possible rules

$$\langle X_i \geq t \rangle \xrightarrow{R_{\geq}} \langle b_{it} \rangle$$

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Generated event: lower bound update

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Constraints with this event

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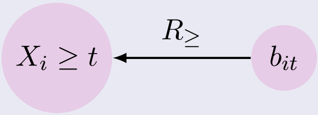
Possible rules

$$\langle X_i \geq t \rangle \xrightarrow{R_{\geq}} \langle b_{it} \rangle$$

Implication graph



Example: Cumulative

Constraints with this event

$$X_i \geq t \iff b_{it}$$

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Possible rules

$$\langle b_{it} \rangle \xrightarrow{R_{\geq}} \langle X_i \geq t \rangle$$

$$\langle b_i \rangle \xrightarrow{R^1_{\land}} \langle b \rangle$$

$$\langle \neg b_i \rangle \xrightarrow{R^2_{\land}} \langle \neg b \rangle \langle b_j \rangle_{\forall j \neq i}$$

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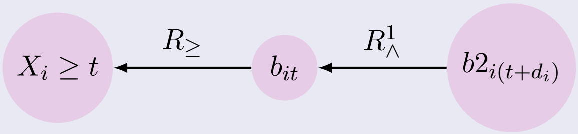

Example: Cumulative

Constraints with this event

$$(b_{i(t-d_i)} \wedge \neg b_{it}) \iff b2_{it}$$

$$\sum_{i \in [\![1,n]\!]} b2_{it} \leq c$$

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$$\langle b \rangle \xrightarrow{R^3_\wedge} \langle b_i \rangle_{\forall i}$$

$$\bot$$

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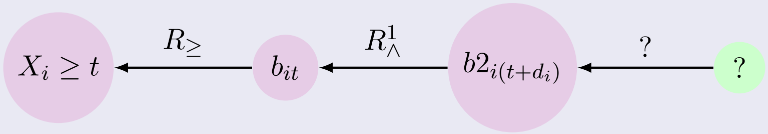

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$$(b_{i(t-d_i)} \wedge \neg b_{it}) \iff b2_{it}$$

Possible rules

$$\frac{b_{it}}{X_i \geq t}[R_{\geq}]$$

$$\frac{b}{b_i}[R_{\wedge}^1] \quad \frac{\neg b \quad b_j \ \forall j \neq i}{\neg b_i}[R_{\wedge}^2]$$

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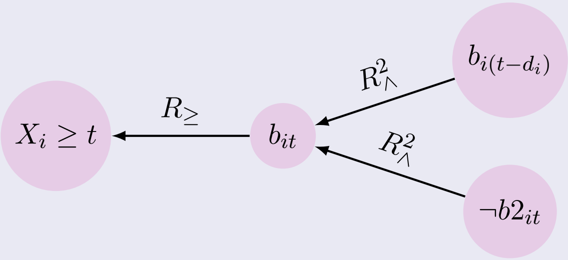
$$\frac{b}{b_i}[R^1_{\land}] \quad \frac{\neg b \quad b_j \ \forall j \neq i}{\neg b_i}[R^2_{\land}]$$

Implication graph



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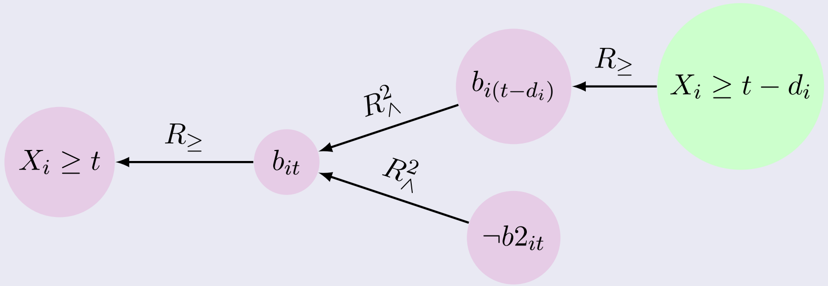
Implication graph



Example: Cumulative

Constraints with this event

$$(b_{i(t-d_i)} \wedge \neg b_{it}) \iff b2_{it}$$

$$\sum_{i \in [\![1,n]\!]} b2_{it} \leq c$$

Example: Cumulative

Constraints with this event

$$(b_{i(t-d_i)} \wedge \neg b_{it}) \iff b2_{it}$$

$$\sum_{i \in [\![1,n]\!]} b2_{it} \leq c$$

Possible rules

$$\frac{\neg b_i \quad \exists i}{\neg b}[R^4_\wedge]$$

$$\frac{b \quad b_j \; \forall j \neq i}{\neg b_i}[R^{inf2}_{sum}]$$

Example: Cumulative

Constraints with this event

$$(b_{i(t-d_i)} \wedge \neg b_{it}) \iff b2_{it}$$

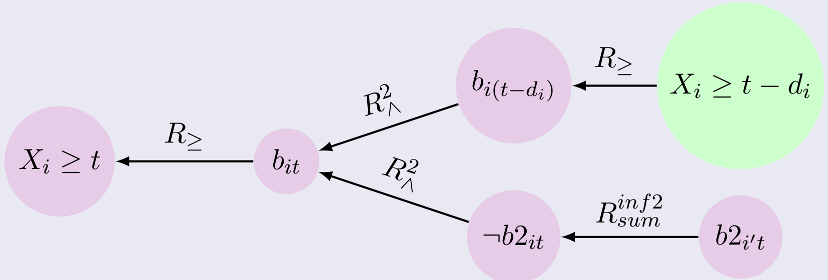
$$\sum_{i \in [\![1,n]\!]} b2_{it} \leq c$$

Possible rules

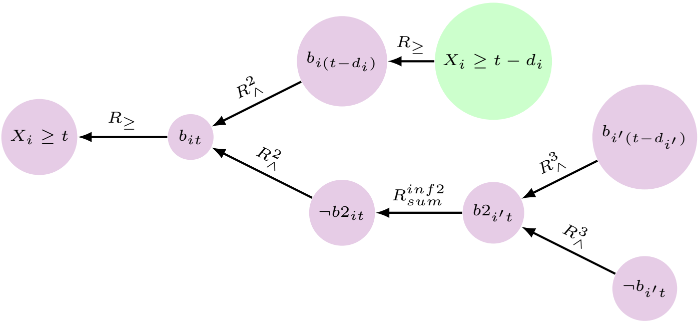
$$\frac{\neg b_i \quad \exists i}{\neg b}[R_\wedge^4]$$

$$\frac{b \quad b_j \ \forall j \neq i}{\neg b_i}[R_{sum}^{inf2}]$$

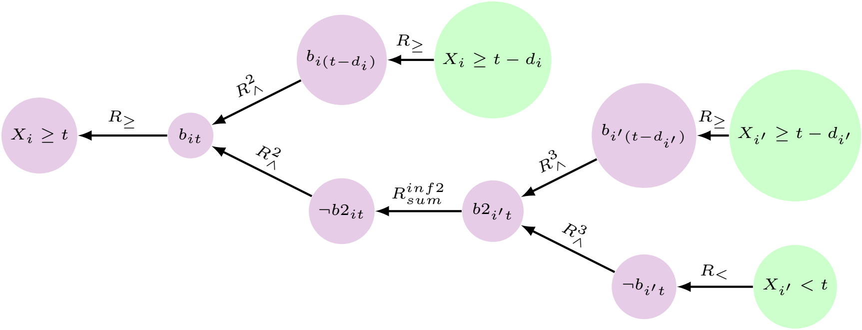
Implication graph



Example: Cumulative



Example: Cumulative



Example: Cumulative

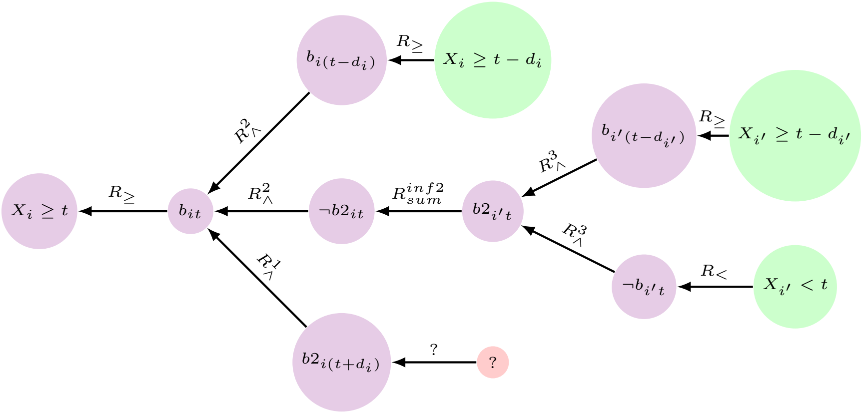


Figure: Lower bound modification event explanation

Example: Cumulative

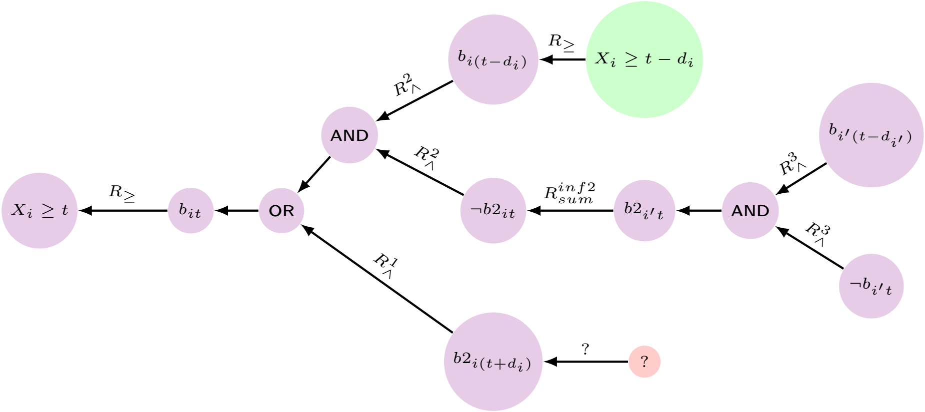


Figure: Lower bound modification event explanation

Generated rule: LaTeX

Red events: Variables saturating time t
Blue events: Ensure Domain coherence

$$\frac{X_i \geq t - d_i \quad X_{i'} \geq t - d_{i'} \quad X_{i'} < t \quad \forall i' \neq i,\ i' \in [\![1, n]\!]}{X_i \geq t}$$

Generated rule: LaTeX

Red events: Variables saturating time t

Blue events: Ensure Domain coherence

$$\frac{X_i \geq t - d_i \qquad X_{i'} \geq t - d_{i'} \qquad X_{i'} < t \qquad \forall i' \neq i,\ i' \in [\![1, n]\!]}{X_i \geq t}$$

$$\frac{X_i < t + d_i \qquad X_{i'} \geq t + d_i - d_{i'} \qquad X_{i'} < t + d_i \qquad \forall i' \neq i,\ i' \in [\![1, n]\!]}{X_i < t}$$